Let $s(t)$ be the length of the curve $y = \cosh^{-1} x$ on the interval $[1, t]$. Find $s'(t)$. **SCORE: _____ / 3 PTS**

$\frac{1}{2}$ $x = \cosh y$ $\quad$ 1 $\quad$ $y \in [0, \cosh^{-1} t]$

$$s'(t) = \frac{d}{dt} \int_0^{\cosh^{-1} t} \sqrt{1 + \sinh^2 y}\, dy = \frac{d}{dt} \int_0^{\cosh^{-1} t} \cosh t\, dt$$

$$= \cosh(\cosh^{-1} t) \cdot \frac{d}{dt} \cosh^{-1} t = \frac{t}{\sqrt{t^2 - 1}}$$

$\frac{1}{2}$ $\quad$ 1

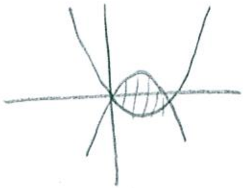
Find the center of mass of the region between the curves $y = 2x^2 - 7x$ and $y = 5x - 2x^2$. **SCORE: _____ / 14 PTS**

$2x^2 - 7x = 5x - 2x^2$

$4x^2 - 12x = 0$

$4x(x-3) = 0$

$x = 0, 3$



2 1

$A = \int_0^3 (5x - 2x^2 - (2x^2 - 7x))\,dx = \int_0^3 (12x - 4x^2)\,dx$

1 $= \left(6x^2 - \frac{4}{3}x^3\right)\Big|_0^3$

$= 54 - 36 = 18$ 1

2

$\bar{x} = \frac{1}{18}\int_0^3 x(5x - 2x^2 - (2x^2 - 7x))\,dx = \frac{1}{18}\int_0^3 (12x^2 - 4x^3)\,dx$

$= \frac{1}{18}(4x^3 - x^4)\Big|_0^3$ 1

$= \frac{1}{18}(108 - 81) = \frac{27}{18} = \frac{3}{2}$ 1

2

$\bar{y} = \frac{1}{18}\int_0^3 \frac{1}{2}\left((5x - 2x^2)^2 - (2x^2 - 7x)^2\right)dx = \frac{1}{36}\int_0^3 (8x^3 - 24x^2)\,dx$

$= \frac{1}{36}(2x^4 - 8x^3)\Big|_0^3$ 1

$= \frac{1}{36}(162 - 216) = -\frac{54}{36} = -\frac{3}{2}$

1

CENTER OF MASS $= \left(\frac{3}{2}, -\frac{3}{2}\right)$ 1

Find the length of the curve $y = \frac{1}{2}\ln(\sec x) + \frac{1}{2}\ln(\csc x)$ **SCORE: _____ / 8 PTS**

between the points $(\frac{\pi}{4}, \frac{1}{2}\ln 2)$ and $(\frac{\pi}{3}, \ln 2 - \frac{1}{4}\ln 3)$.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1 + \left(\frac{1}{2}\frac{\sec x \tan x}{\sec x} - \frac{1}{2}\frac{\csc x \cot x}{\csc x}\right)^2}\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1 + \left(\underbrace{\tfrac{1}{2}\tan x - \tfrac{1}{2}\cot x}_{}\right)^2}\, dx \quad 2$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{1 + \underbrace{\tfrac{1}{4}\tan^2 x - \tfrac{1}{2} + \tfrac{1}{4}\cot^2 x}_{}}\, dx \quad 2$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \sqrt{\tfrac{1}{4}\tan^2 x + \tfrac{1}{2} + \tfrac{1}{4}\cot^2 x}\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \underbrace{\left(\tfrac{1}{2}\tan x + \tfrac{1}{2}\cot x\right)}_{}\, dx \quad 1\tfrac{1}{2}$$

$$= \underbrace{\left(\tfrac{1}{2}\ln(\sec x) - \tfrac{1}{2}\ln(\csc x)\right)}_{}\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}} \quad 1\tfrac{1}{2}$$

$$= \underbrace{\tfrac{1}{2}\ln 2 - \tfrac{1}{2}\ln\tfrac{2}{\sqrt{3}}}_{} \; 1 = \tfrac{1}{2}\ln\sqrt{3} = \underbrace{\tfrac{1}{4}\ln 3}_{} \quad 1 \text{ BONUS POINT}$$

The time needed for a certain search engine to respond to a request is a continuous random variable X **SCORE: _____ / 8 PTS**

(in units of tenths of a second) with probability density function $f(x)=\begin{cases} k\sqrt[3]{9-x}, & x\in[1,9] \\ 0, & x\notin[1,9] \end{cases}$ for some constant k.

Find the median response time.

$$\int_1^9 k(9-x)^{\frac{1}{3}}\,dx = 1 \quad 2$$

$$u = 9-x \qquad x=9 \to u=0$$
$$du = -dx \qquad x=1 \to u=8$$
$$x = x_{MEDIAN} \to u = 9-x_{MEDIAN}$$

$$-k\int_8^0 u^{\frac{1}{3}}\,du = 1$$

$$-k\left(\tfrac{3}{4}u^{\frac{4}{3}}\right)\Big|_8^0 = 1 \quad 1$$

$$-k(-12) = 1$$

$$k = \tfrac{1}{12} \quad 1$$

$$\int_1^{x_{MEDIAN}} \tfrac{1}{12}(9-x)^{\frac{1}{3}}\,dx = \tfrac{1}{2} \quad 2$$

$$-\tfrac{1}{12}\int_8^{9-x_{MEDIAN}} u^{\frac{1}{3}}\,du = \tfrac{1}{2}$$

$$\tfrac{3}{4}u^{\frac{4}{3}}\Big|_8^{9-x_{MEDIAN}} = -6$$

$$1 \quad \tfrac{3}{4}\left[(9-x_{MEDIAN})^{\frac{4}{3}} - 16\right] = -6$$

$$(9-x_{MEDIAN})^{\frac{4}{3}} = 8$$

$$1 \quad x_{MEDIAN} = 9 - 8^{\frac{3}{4}}$$